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Rainbow graphs and semigroups

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We give an algebraic characterization of rainbow graphs. A connected graph Γ *is called rainbow if there is a vertex coloring of* Γ*, which is bijective on the set of neighbors of each vertex of* Γ.

A *rainbow graph* [3] is a connected graph Γ with the set of vertices $V(\Gamma)$ and the set of edges $E(\Gamma)$ that can be vertex-colored $\chi : V(\Gamma) \longrightarrow \kappa$ so that every color $x \in \kappa$ is represented once, and only once, among the neighbors $N(v) = \{u \in V(\Gamma)\colon \{u, v\} \in E(\Gamma)\}$ of each vertex $v \in V(\Gamma)$. For applications of rainbow graphs, see [1]. If one removes the edge-matching of the monochrome edges of a rainbow graph, one gets a kaleidoscopical graph [2, Chapter 6].

Let κ be a cardinal. A *rainbow semigroup* $RS(\kappa)$ is a semigroup in the alphabet κ determined by the relations $xxx = x$, $xyx = x$ for all $x, y \in \kappa$. We identify $RS(\kappa)$ with the set of all non-empty words in κ with no factors xxx, xyx.

For $x \in \kappa$, a *rainbow group* $RG(\kappa, x)$ is a subset of $RS(\kappa)$ containing x and all words of the form xwx, $w \in RS(\kappa)$. The word xx is the identity of $RG(\kappa, x)$, $x^{-1} = x$, and $(xwx)^{-1} = xx\widetilde{w}xx$, where $\widetilde{w}$ is the word w written in the reverse order.

Theorem 1. *For any cardinal* κ *and each* $x \in \kappa$*, the following statements hold:*

(*i*) *the idempotents of* $RS(\kappa)$ *are only* yz*, where* y, $z \in \kappa$*;*

(*ii*) $RG(\kappa, x)$ *is a free product of the cyclic group* $\langle x \rangle$ *of order 2 and the family of infinite cyclic groups* $\{\langle xabx \rangle\colon a, b \in \kappa, a \neq x, b \neq x\}$*;*

(*iii*) $RS(\kappa)$ *is a sandwich product* $RS(\kappa) = L(x) \times RG(\kappa, x) \times R(x)$*, where* $L(x) = \{yx\colon y \in \in \kappa\}$*,* $R(x) = \{xy\colon y \in \kappa\}$*, and the multiplication* $(l_1, w_1, r_1)(l_2, w_2, r_2) = (l_1, w_1 r_1 l_2 w_2, r_2)$*.*

Let κ be a cardinal, $x \in \kappa$. An equivalence $\sim$ on $RS(\kappa)$ is called a *rainbow equivalence* if, for any w_1, $w_2 \in RS(\kappa)$, we have

- $w_1 \sim w_2 \Longrightarrow l(w_1) = l(w_2)$, where $l(w)$ is the first letter of w;
- $w_1 \sim w_2 \Longrightarrow yw_1 = yw_2$ for each $y \in \kappa$;
- $l(w) = y \Longrightarrow w$ and yw are not equivalent;
- $w \sim wxx$ for each $w \in KS(\kappa)$.

Each rainbow equivalence $\sim$ on $RS(\kappa)$ determines the rainbow graph $\Gamma(\kappa, k)$ as follows. The set $V(\Gamma)$ of vertices of Γ is a factor-set $RS(\kappa)/\sim = \{[w]\colon w \in RS(\kappa)\}$, where $[w]$ is the class of equivalence $\sim$ containing w. By definition, $\{u, v\} \in E(\Gamma)$ if and only if $u \neq v$, and there exists $w \in u$ such that $yw \in v$. Then the mapping $\chi\colon V(\Gamma) \longrightarrow \kappa$ defined by $\chi([w]) = l(w)$ does not depend on the choice of w and determines a rainbow coloring of Γ.

In turn, every rainbow equivalence $\sim$ on $RS(\kappa)$ is uniquely determined by the subgroup

$$S_x = [xx] \bigcap RG(\kappa, x)$$

of $RG(\kappa, x)$ because

$$w_1 \sim w_2 \Longleftrightarrow l(w_1) = l(w_2) \wedge xw_1xx \sim xw_2xx \Longleftrightarrow (xw_1xx)^{-1}(xw_2xx) \in S_x.$$

We say that two rainbow graphs Γ_1, Γ_2 with rainbow colorings $\chi_1\colon V(\Gamma_1) \longrightarrow \kappa$, $\chi_2\colon V(\Gamma_2) \longrightarrow \kappa$ are rainbow isomorphic if there exists a bijection $f\colon V(\Gamma_1) \longrightarrow V(\Gamma_2)$ such that

- $\forall\, u,\, v \in V(\Gamma_1)\colon \{u, v\} \in E(\Gamma_1) \iff \{f(u), f(v)\} \in E(\Gamma_2)$;
- $\forall\, u \in V(\Gamma_1)\colon \chi_1(u) = \chi_2(f(u))$.

Now, we are ready to characterize all rainbow graphs up to rainbow isomorphisms.

Let Γ be a rainbow graph with rainbow coloring $\chi\colon V(\Gamma) \longrightarrow \kappa$. We define a transitive action of $RS(\kappa)$ on the set $V(\Gamma)$ as follows. Let $v \in V(\Gamma)$, $x \in \kappa$. Pick $u \in N(v)$ such that $\chi(u) = x$ and put $x(v) = u$. Then we extend the action onto $KS(\kappa)$ inductively. If $w = RS(\kappa)$, $w = xw'$, $x \in \kappa$, we put $w(v) = \chi(w'(v))$. Given any $v_1, v_2 \in V(\Gamma)$, the sequence of colors of the vertices on a path from v_1 to v_2 determines a word $w \in RS(\kappa)$ such that $w(v_1) = v_2$, so $RS(\kappa)$ acts on $V(\Gamma)$ transitively. Clearly, the group $RG(\kappa, x)$ acts transitively on the set of vertices of color x.

We fix $v \in V(\Gamma)$ with $\chi(v) = x$, determine a rainbow equivalence $\sim$ on $RS(\kappa)$ by the rule

$$w \sim w' \iff w(v) = w'(v),$$

and note that the graphs Γ and $\Gamma(\kappa, \sim)$ are rainbow isomorphic via the bijection $f\colon V(\Gamma) \longrightarrow KS(\kappa)/\sim$, $f(u) = \{w \in KS(\kappa)\colon w(v) = u\}$.

Thus, we get the following statement.

Theorem 2. *For every rainbow graph Γ with rainbow coloring $\chi\colon V(\Gamma) \longrightarrow \kappa$, there exists a rainbow equivalence $\sim$ on $RS(\kappa)$ such that Γ and $\Gamma(\kappa, \sim)$ are rainbow isomorphic. Every rainbow equivalence on $RS(\kappa)$ is uniquely determined by some subgroup of $RG(\kappa, x)$.*

Let $\Gamma(V, E)$ be a connected graph with the set of vertices V, and let the set of edges E, d be the path metric on V, $B(v, r) = \{u \in V\colon d(v, u) \leqslant r\}$, $v \in V$, $r \in \omega = \{0, 1, \ldots\}$.

A graph $\Gamma(V, E)$ is called *kaleidoscopical* [6] if there exists a coloring (a surjective mapping) $\chi\colon V \longrightarrow \kappa$, κ is a cardinal such that the restriction $\chi \mid B(v, 1)\colon B(v, 1) \longrightarrow \kappa$ is a bijection on each unit ball $B(v, 1)$, $v \in V$. For kaleidoscopical graphs, see also [2, Chapter 6] and [5].

Let G be a group, and let X be a transitive G-space with the action $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. A subset A of X, $|A| = \kappa$ is said to be a *kaleidoscopical configuration* [4] if there exists a coloring $\chi\colon X \longrightarrow \kappa$ such that, for each $g \in G$, the restriction $\chi \mid gA\colon gA \longrightarrow \kappa$ is a bijection.

We note that kaleidoscopical graphs and kaleidoscopical configurations can be considered as partial cases of kaleidoscopical hypergraphs defined in [2, p.5]. Recall that a *hypergraph* is a pair $(X, \mathfrak{F})$, where X is a set, $\mathfrak{F}$ is a family of subsets of X.

A hypergraph $(X, \mathfrak{F})$ is said to be *kaleidoscopical* if there exists a coloring $\chi\colon X \longrightarrow \kappa$ such that, for each $F \in \mathfrak{F}$, the restriction $\chi \mid F\colon F \longrightarrow \kappa$ is a bijection.

Clearly, a graph $\Gamma(V, E)$ is kaleidoscopical if and only if the hypergraph $(V, \{B(v, 1)\colon v \in V\})$ is kaleidoscopical. A subset A of a G-space X is kaleidoscopical if and only if the hypergraph $(X, \{g(A)\colon g \in G\})$ is kaleidoscopical.

We say that two hypergraphs $(X_1, \mathfrak{F}_1)$, $(X_2, \mathfrak{F}_2)$ with kaleidoscopical colorings $\chi_1\colon X_1 \longrightarrow \kappa$, $\chi_2\colon X_2 \longrightarrow \kappa$ are *kaleidoscopically isomorphic* if there is a bijection $f\colon X_1 \longrightarrow X_2$ such that

- $\forall\, A \subseteq X_1\colon A \in \mathfrak{F}_1 \iff f(A) \in \mathfrak{F}_2$;
- $\forall\, x \in X_1\colon \chi_1(x) = \chi_2(f(x))$.

We describe an algebraic construction which gives all kaleidoscopical graphs up to isomorphisms.

The *kaleidoscopical semigroup* $KS(\kappa)$ is a semigroup in the alphabet κ determined by the relations $xx = x$, $xyx = x$ for all $x, y \in \kappa$. For our purposes, it is convenient to identify $KS(\kappa)$ with the set of all non-empty words in κ with no factors xx, xyx, where $x, y \in \kappa$.

For every $x \in \kappa$, the set $KG(\kappa, x)$ of all words from $KS(\kappa)$ with the first and the last letter x is a subgroup (with the identity x) of the semigroup $KS(\kappa)$. To obtain the inverse element to the word $w \in KG(\kappa, x)$, it suffices to write w in the inverse order. The group $KG(\kappa, x)$ is called the *kaleidoscopical group* in the alphabet κ with the identity x.

For finite cardinals κ, the following theorem is proved in [2, pp. 64–66]: but corresponding arguments work for arbitrary κ.

Theorem 3. *For any cardinal κ, the following statements hold:*

(i) idempotents of the semigroup $KS(\kappa)$ are the only words x, xy, where $x, y \in \kappa$, $x \neq y$,

(ii) the kaleidoscopical group $KG(k, x)$ is a free group with the set of free generators

$$\{xyzx\colon y, z \in \kappa \setminus \{x\}, y \neq z\},$$

(iii) the kaleidoscopical semigroup $KS(\kappa)$ is isomorphic to the sandwich product $L(x) \times$ $\times\, KG(\kappa, x) \times R(x)$ with the multiplication

$$(l_1, g_1, r_1)(l_2, g_2, r_2) = (l_1, g_1 r_1 l_2 g_2, r_2),$$

where $L(x) = \{yx\colon y \in \kappa\}$, $R(x) = \{xy\colon y \in \kappa\}$.

We fix $x \in \kappa$, denote the first letter of the word $w \in KS(\kappa)$ by æ(w), and say that an equivalence $\sim$ on $KS(\kappa)$ is *kaleidoscopical* if, for all $w, w' \in KS(\kappa)$ and $y \in \kappa$,

$$w \sim w' \Longrightarrow \text{æ}(w) = \text{æ}(w') \wedge yw = yw',$$

$$w \sim w' \Longleftrightarrow wx \sim w'x.$$

Let $[w]$ be the class of equivalence $\sim$ containing $w \in KS(\kappa)$.

We put

$$S_x = [x] \bigcap KG(\kappa, x),$$

observe that S_x is a subgroup of $KG(\kappa, x)$, and show that $\sim$ is uniquely determined by S_x:

$$w \sim w' \Longleftrightarrow \text{æ}(w) = \text{æ}(w') \wedge xwx \sim xw'x \Longleftrightarrow (xwx)^{-1}(xw'x) \in S_x.$$

We see also that any subgroup of $KG(\kappa, x)$ can be taken as S_x to determine a kaleidoscopical equivalence on $KS(\kappa)$.

A kaleidoscopical equivalence $\sim$ determines a graph $\Gamma(\kappa, \sim)$ with the set of vertices $KS(\kappa)/\sim$ and the set of edges E defined by the rule:

$$(u, v) \in E \Longleftrightarrow \exists\, w \in u \exists\, y \in \kappa\colon \text{æ}(w) \neq y \wedge yw \in v.$$

A coloring $\chi\colon KS(\kappa)/\sim \longrightarrow \kappa$ defined by $\chi([w]) = \text{æ}(w)$ shows that $\Gamma(\kappa, \sim)$ is kaleidoscopical.

Now let $\Gamma(V, E)$ be a kaleidoscopical graph with kaleidoscopical coloring $\chi\colon V \longrightarrow \kappa$. We define a transitive action of the semigroup $KS(\kappa)$ on the set V as follows. Let $v \in V$, $x \in \kappa$. Pick $u \in B(v, 1)$ such that $\chi(u) = x$ and put $x(v) = u$. Then we extend the action onto $KS(\kappa)$ inductively. If $w = KS(\kappa)$, $w = xw'$, $w' \in KS(\kappa)$, $x \in \kappa$, we put $w(v) = x(w'(v))$. Given any $v_1, v_2 \in V$, the sequence of colors of the vertices on a path from v_1 to v_2 determines a word $w \in KS(\kappa)$ such that $w(v_1) = v_2$, so $KS(\kappa)$ acts on V transitively. Clearly, the group $KG(\kappa, x)$ acts transitively on the set $\chi^{-1}(x)$ of vertices of color x.

We fix $v \in V$ with $\chi(v) = x$, determine a kaleidoscopical equivalence $\sim$ on $KS(\kappa)$ by the rule

$$w \sim w' \Longleftrightarrow w(v) = w'(v),$$

and note that the graphs $\Gamma(V, E)$ and $\Gamma(\kappa, \sim)$ are kaleidoscopically isomorphic via the bijection $f\colon V \longrightarrow KS(\kappa)/\sim$, $f(u) = \{w \in KS(\kappa)\colon w(v) = u\}$.

All above considerations are focused in the following theorem.

Theorem 4. *For every kaleidoscopical graph $\Gamma(V, E)$ with kaleidoscopical coloring $\chi\colon V \longrightarrow \kappa$, there exists a kaleidoscopical equivalence $\sim$ on the semigroup $KS(\kappa)$ such that $\Gamma(V, E)$ is kaleidoscopically isomorphic to $\Gamma(\kappa, \sim)$. Every kaleidoscopical equivalence $\sim$ on $KS(\kappa)$ is uniquely determined by some subgroup of the group $KG(\kappa, x)$.*

Every group G can be considered as a G-space with the left regular action $(g, x) \longmapsto yx$. Let A be a kaleidoscopical configuration in G. By [4, Corollary 1.3], A is complemented, i. e. there exists a subset B of G such that the multiplication $A \times B \longrightarrow G$, $(a, b) \longmapsto ab$ is bijective.

Let A be a system of generators of a group G such that $A = A^{-1}$ and $e \in A$, e is the identity of G. We consider the Cayley graph $Cay(G, A)$ with the set of vertices G and the set of edges E defined by the rule:

$$(g, h) \in E \Longleftrightarrow g^{-1}h \in A, g \neq h.$$

Clearly, $Cay(G, A)$ is connected. Assume that $Cay(G, A)$ is kaleidoscopical with kaleidoscopical coloring $\chi\colon G \longrightarrow |A|$. Since $B(g, 1) = gA$ and χ is bijective on each ball $B(g, 1)$, we see that A is a kaleidoscopical configuration. On the other hand, if A is a kaleidoscopical configuration in G with kaleidoscopical coloring $\chi\colon G \longrightarrow A$, then χ is bijective on each set gA. So, $Cay(G, A)$ is kaleidoscopical. Thus, we get the following theorem.

Theorem 5. *Let G be a group, and let A be a system of generators of G such that $A = A^{-1}$ and $e \in A$. Then A is a kaleidoscopical configuration if and only if $Cay(G, A)$ is kaleidoscopical.*

We conclude the paper with two open questions.

Question 1. *How can one detect whether a kaleidoscopical hypergraph is kaleidoscopically isomorphic to a hypergraph of unit balls of some kaleidoscopical graph?*

Question 2. *How can one detect whether a kaleidoscopical hypergraph is kaleidoscopically isomorphic to a hypergraph determined by a kaleidoscopical configuration in a G-space?*

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Веселкові графи і напівгрупи

Отримано алгебраїчну характеризацію веселкових графів. Зв'язний граф Γ *називаєтьcя веселковим, якщо існує розфарбування множини вершин* Γ*, що є бієктивним на множині сусідів кожної вершини* Γ*.*

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Радужные графы и полугруппы

Получена алгебраическая характеризация радужных графов. Связный граф Γ *называется радужным, если существует раскраска множества вершин* Γ*, биективная на множестве соседей для каждой вершины* Γ*.*